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Fluctuation theorem for a deterministic one-particle system

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A Duffing oscillator is driven by a sum of N chaotic time series. These time series are solutions of the undriven Duffing equation. It is shown that $N=1$ is sufficient to render the fluctuation theorem [Gallavotti and Cohen, Phys. Rev. Lett. **74**, 2694 (1995); Gallavotti, J. Math. Phys. **41**, 4061 (2000); Evans and Searles, Adv. Phys. **51**, 1529 (2002)] for the power J_τ averaged within intervals of length τ. In particular, the probabilities $p(J_\tau)$ follow a nearly Gaussian distribution. Also, $\ln[p(J_\tau)/p(-J_\tau)]$ versus J_τ can be fitted by strikingly linear functions, the slopes being proportional to τ for large τ. These results indicate that validity of the fluctuation theorem requires neither a many-particle system nor a stochastic process, which are requirements used in previous works.

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I. INTRODUCTION

The fluctuation theorem (FT) derived by Gallavotti and Cohen [1] has been considered in literature as a generalization of the second law of thermodynamics for finite systems. (See also the reviews [2,3].) One of its formulations is

$$\frac{p(J_\tau)}{p(-J_\tau)} = e^{J_\tau \tau/\beta_\tau}, \tag{1}$$

where $p(J_\tau)$ is the probability of the mean flux J_τ (of heat, momentum, work, etc.) during the time τ. Here, we consider the mean flux of the work W_τ, i.e., the mean power $J_\tau = W_\tau/\tau$: $-W_\tau$ is the work performed by the system within τ. β_τ approaches a constant value β_∞ as $\tau \to \infty$. β_∞ has been considered as an effective "temperature" of a system out of equilibrium (see e.g., [4]).

The FT was proven by considering time reversible (highly) chaotic Anosov systems consisting of many particles. The theorem had been shown to hold for shear-flow simulations [5]. An experimental verification was reported for turbulent Rayleigh-Benard flow [6]. Further experimental evidence was given by following the trajectory of a Brownian particle captured in an optical trap that is translated relative to the surrounding water molecules [7,8]. Numerical verification of the FT was achieved by simulations of chains of coupled nonlinear oscillators [9,10], of an electrical conduction device [11], of the shell model of turbulence [4], and of the Burridge-Knopoff earthquake model consisting of chains of blocks connected by springs [4].

The FT has been considered as a bridge between time-reversible microscopic equations of mechanics and the time-irreversible macroscopic equations of thermodynamics [8]. In this context, it is remarkable that there are versions of the FT which also hold for stochastic processes certainly not being described by reversible trajectories. One example is a FT formulated by Kurchan for certain diffusion processes [12]. That FT was extended to general Markov processes by Lebowitz and Cohen [13]. Recently, Gaspard [14] derived a FT in the framework of the master equation by Nicolis *et al.* (see references in [14]), which was applied to describe jumps in a bistable chemical reaction. Fluctuation theorems describing a dragged Brownian particle [15] and electric circuits [16] were also derived from stochastic equations. Discussions on the role of stochasticity, as compared to that of microscopic reversibility, are given in [8,13,17] and references therein.

In all investigated formulations of the FT so far, the source of disorder has been a large number of particles (or elements in a chain), or a stochastic process [12–16]. In contrast, we will consider here a system S consisting of a single deterministically chaotic particle with three degrees of freedom. S is driven chaotically by a signal obtained from the undriven S.

II. THE MODEL

We consider a driven chaotic oscillator with a single particle, described by

$$m\ddot{x} + \alpha\dot{x} = F(x,t) + \frac{1}{N}\sum_{n=1}^{N} \xi\ddot{\tilde{x}}_n. \tag{2}$$

The force $F(x,t)$ is that of a Duffing oscillator, i.e.,

$$F(x,t) = A\cos(\omega t) - \partial V/\partial x, \tag{3}$$

$$V = x^4/4 - x^2/2. \tag{4}$$

Setting $m=1$, $\alpha=0.25$, $A=0.38$, $\omega=1$, and $\xi=0$, this oscillator is chaotic. For the driving of this chaotic system, we set

$$\tilde{x}_n(t) = x_{\xi=0}(t-\phi_n). \tag{5}$$

In other words, the $\tilde{x}_n$ are obtained by integration of Eqs. (2)–(4) with $\xi=0$. The phase shifts ϕ_n are introduced in order to avoid the $\tilde{x}_n$ being correlated with each other and with x. We set $\phi_n = nT$, where $T=10^6$. This avoidance of correlations could as well be accomplished with $\phi_n=0$ by a proper choice of different initial conditions for the $\tilde{x}_n$ and x.

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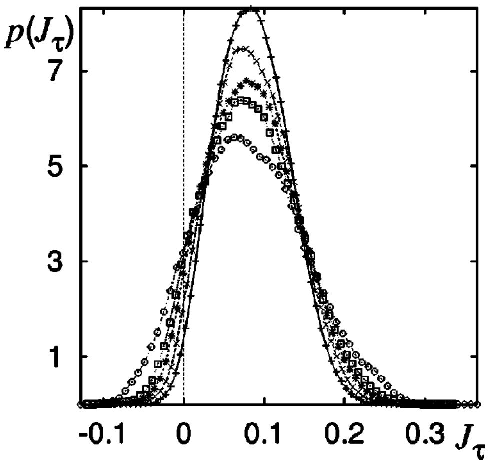

FIG. 1. Probability distribution function of the mean power J_τ. $\tau=16.3$ (○), 20.8 (□), 23.8 (*), 28.2 (×), and 34.1 (+).

If not stated otherwise, driving was performed with $\xi=1$. After allowing transients to die away, we set the total time to 1.4×10^7 intervals of length τ and determined the mean power J_τ in each interval by

$$J_\tau=\frac{1}{\tau}\int_t^{t+\tau}F\dot{x}dt. \tag{6}$$

Integration was performed with a Gear predictor-corrector algorithm of fifth order with an integration time step of $\Delta t=10^{-4}$.

III. RESULTS AND DISCUSSION

Setting $N=1$, i.e., assuming only one chaotic driving term in Eq. (2), we obtained the probability distributions of J_τ shown in Fig. 1. Here, $J_\tau<0$ signifies work performed by the system within the time τ. The negative parts of the curves clearly become smaller as τ increases, leading to the well-known "classical" second law of thermodynamics. The distributions in Fig. 1 yielded plots of $\ln[p(J_\tau)/p(-J_\tau)]$ versus J_τ, which are exemplified in Fig. 2(a). We obtained strikingly linear relationships, as in the original formulation of the FT [1–3].

According to Eq. (1), the slopes S_τ of fitted straight lines [exemplified in Fig. 2(a)] should be given by $S_\tau=\tau/\beta_\tau$. We thus plotted $\beta_\tau=\tau/S_\tau$ versus τ. This is shown in Fig. 2(b), where we see a monotonically decreasing dependence $\beta_\tau(\tau)$, saturating at large τ to an effective "temperature" β_∞, as has also been obtained in *RC* circuits with noise [16], in a turbulent fluid [6], in the shell model of turbulence, and in the Burridge-Knopoff earthquake model [4]. A constant β_τ for large τ also appears in the FT in its original form [1–3].

In Fig. 2(c) we show the effective "temperature" β_∞ as a function of the driving amplitude ξ. We clearly see that β_∞ increases as one increases the energy injected into the system. At least from that point of view, the definition of a "temperature" makes some sense, although we are dealing here with a dissipative system out of equilibrium consisting of only one particle.

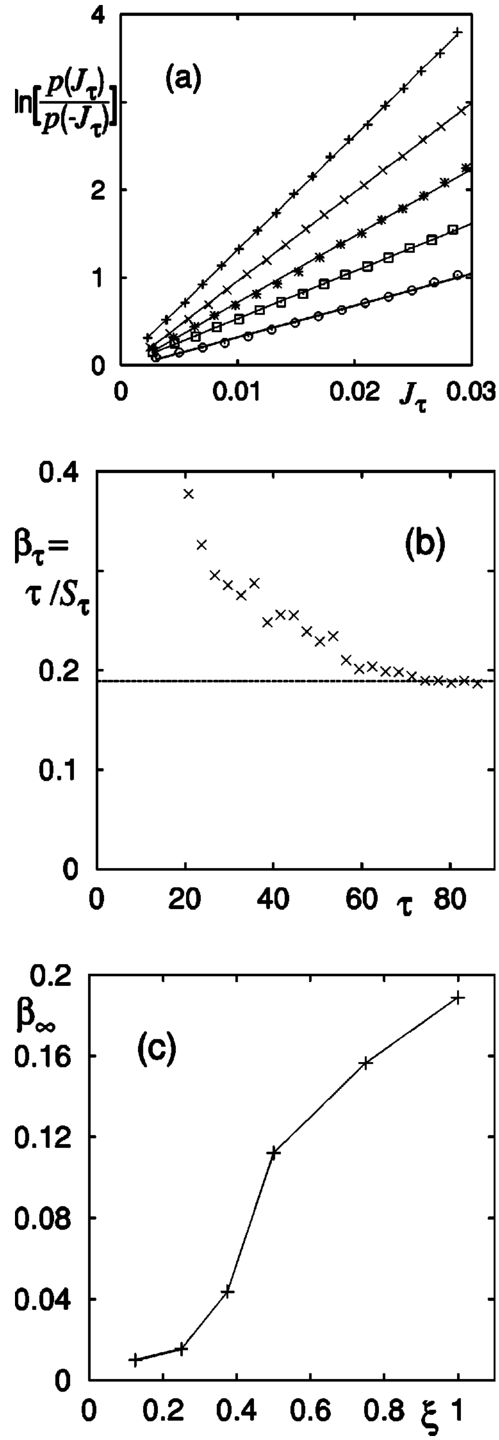

FIG. 2. (a) $\ln[p(J_\tau)/p(-J_\tau)]$ vs J_τ, as obtained from the probability distributions shown in Fig. 1. (b) $\beta_\tau=\tau/S_\tau$ vs τ. The S_τ are the slopes of straight lines fitted to plots exemplified in (a). β_τ reaches a saturating value for large τ. (c) Saturating value β_∞ (β_τ for large τ) vs the amplitude ξ of the chaotic driving with $N=1$.

In Fig. 3(b), we show probability distributions which are normalized by plotting $p(J_\tau)\sigma_\tau$ versus $\tilde{J}_\tau=(J_\tau-\langle J_\tau\rangle)/\sigma_\tau$, where σ_τ is the standard deviation of J_τ. Figure 3(b) shows that these normalized distributions are independent of τ. Furthermore, they roughly follow a Gaussian distribution, as has

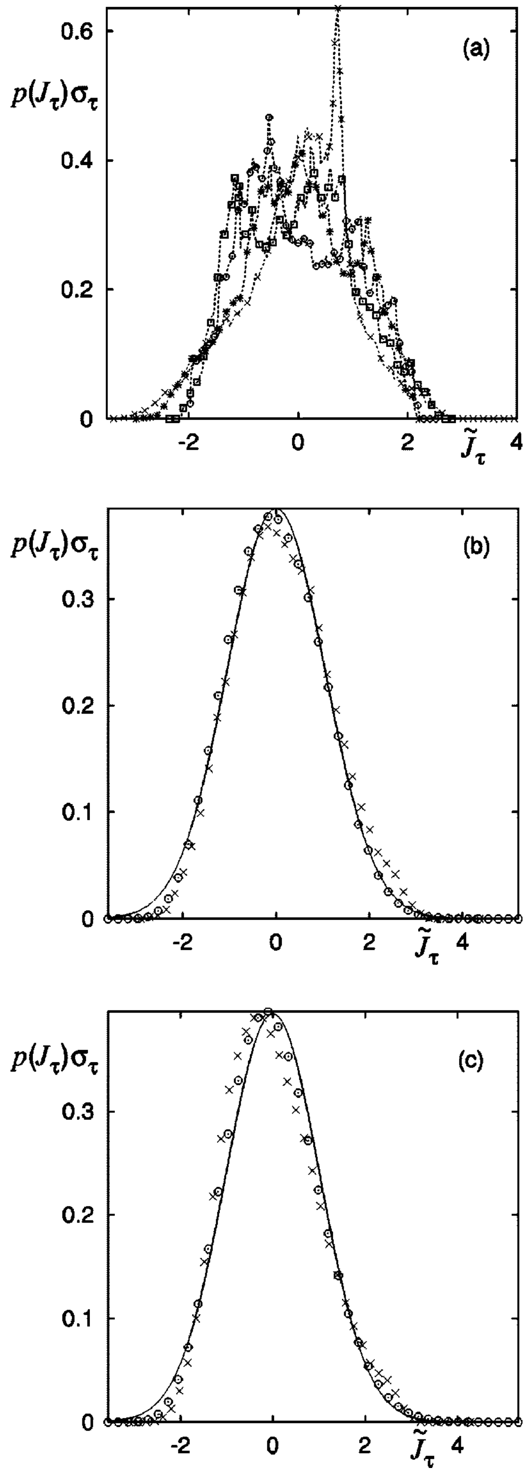

FIG. 3. Normalized probability function $p(J_\tau)\sigma_\tau$ vs the reduced variable $\tilde{J}_\tau=(J_\tau-\langle J_\tau\rangle)/\sigma_\tau$. σ_τ: standard deviation of J_τ (a) $\xi=0$; $\tau=7.1$ (○), 13.4 (□), 19.6 (*), and 25.8 (×). (b) and (c) $\tau=16.3$ (×) and 43.1 (○); the functions roughly collapse into a Gaussian curve (full line) with zero mean and unit standard deviation. (b) $N=1$. (c) $N=2$.

been reported for other simulations and for experiments [4,6,15].

The recent development of "superstatistics" [18,19], accounting for fluctuations of the effective temperature in a Gaussian distribution leads us to correct our fit to the points for $\tau=43.1$ in Fig. 3(b) (circles). We did so by assuming a quadratic correction term (see [18]) of the probability function, obtaining a reduction of the sum of squares of residuals from 0.018 [no correction; full line in Fig. 3(b)] to 0.012 (correction with an entropic index $q=1.2$). A further improvement of the fit, considering the cubic term and specific types of superstatistics [18], was not possible because of the asymmetry of our probability function owing to limitations of computing time.

So far we assumed $N=1$. It thus remains to examine the effect of different N. For $N=2$, i.e., for two chaotic driving terms in Eq. (2), we obtained the same qualitative behavior as that depicted in Figs. 1 and 2, i.e., for $N=1$. In other words, chaoticity is sufficiently strong with one single uncorrelated chaotic driving. This is exemplified in Fig. 3(c) ($N=2$), as compared with Fig. 3(b) ($N=1$). In contrast, setting $\xi=0$, i.e., assuming no chaotic driving, we do not obtain results comparable to those presented above [see Fig. 3(a)]; these distributions were found to be unaffected by doubling the number of evaluated time intervals of length τ. The irregular peaks in each distribution function in Fig. 3(a) are invariant and clearly related to the dynamical properties of the Duffing equation, while the addition of uncorrelated chaos smoothens out these many peaks, rendering a nearly Gaussian distribution [see Figs. 3(b) and 3(c)].

IV. CONCLUSIONS

In the derivation of the FT in [1], a many-particle, time reversible, chaotic Asonov system was assumed. It had already been demonstrated that the FT is also valid for dissipative systems [4–8]. Furthermore, it was also shown that a many-particle system is not necessary if stochasticity is assumed [12–16].

We could show here that a FT is obtained by considering a single deterministically chaotic particle with three degrees of freedom, which is driven by deterministic chaos. The simplicity of this system is even more striking if one considers that our driving by deterministic chaos is taken from the trajectory of the same particle in the absence of driving. It is left as a future task to find general conditions for a fluctuation theorem describing a single, chaotically oscillating, deterministic particle, such as that investigated here.

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